Computers & Structures Vol. 26, No. 4, pp. 587–596, 1987
Printed in Great Britain.

0045-7949/87 $3.00 + 0.00
Pergamon Journals Ltd

NUMERICAL YIELD LINE ANALYSIS

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(*Received* 11 *August* 1986)

Abstract—A numerical method based on the virtual work approach of the yield line theory is presented. The method consists of computing the yield load of a plate based on the geometry of an assumed collapse mechanism defined by means of nodes, planes and lines. Since the method is numerical, it allows the yield line analysis of plates with complex shapes, assumed mechanisms and loadings. Algorithms for the calculations of the work done by the external loads on the plate and the internal work dissipated by the yield lines in the assumed mechanism are described. The features of a computer program are outlined, and a numerical example of the numerical yield line analysis of a reinforced concrete slab is given.

INTRODUCTION

The yield line method is a simple and efficient method to calculate the plastic collapse load of flat, relatively thin, plates of rigid–perfectly plastic material when transversely loaded in bending. The method was developed largely by Johansen [1] and since then, it has been applied successfully to both concrete and steel plates [2–4].

A numerical method based on the yield line theory is presented in this paper. The method differs from the conventional yield line method in that it does not use a direct algebraic description of the problem but rather it uses analytical geometry, vector algebra and the specific dimensions of the problem on hand to arrive at the solution.

The method presented is general and since it is entirely numerical, it can be applied to plates of arbitrary shape which can be assumed to form any arbitrary yield line mechanism. Furthermore, the method has the advantage of requiring no algebraic manipulations and thus it is not limited by the complexity of the yield line pattern and by the resulting complexity of the algebra, as is sometimes the case with the conventional yield line method.

The yield line theory is briefly reviewed below. However, a basic understanding of the yield line theory is assumed in the following discussion and the reader is referred to the standard texts on the subject [1, 2].

The yield line method is based on the kinematic theorem of the plastic theory of structures and gives an upper bound solution for the collapse load of a plate.

In the yield line method, a plastic collapse mechanism of the plate is assumed consisting of undeformed plate segments connected by plastic hinge lines, usually called yield lines. The mechanism must be kinematically admissible over the whole plate and at the boundaries. The bending moment distribution is not considered and, in general, the equilibrium conditions are not verified.

There are two solution approaches in the yield line theory: the virtual work method and the so-called equilibrium method. Both methods lead to identical upper bound solutions, and it has been demonstrated that both methods represent in fact the same solution, but with a different approach [2]. The virtual work method is simpler in principle and is used for the numerical method presented in this paper. The virtual work method is outlined below.

In this method, a plastic collapse mechanism is assumed for a given plate and loading, and the collapse load P is found by equating the work done by the external loads on the plate to the internal work dissipated by the yield lines during a small motion of the assumed collapse mechanism: namely,

$$E = D \tag{1}$$

i.e.

$$\sum_{\text{all loads}} \lambda P \delta = \sum_{\text{all yield lines}} \gamma m_p \theta l \tag{2}$$

or

$$P = \frac{\sum_{\text{all yield lines}} \gamma m_p \theta l}{\sum_{\text{all loads}} \lambda \delta}; \tag{3}$$

alternatively,

$$m_p = \frac{\sum_{\text{all loads}} \lambda P \delta}{\sum_{\text{all yield lines}} \gamma \theta l}, \tag{4}$$

where λP is an applied load of magnitude λ acting through a virtual displacement δ. γm_p is the plastic moment resistance per unit length of magnitude γ, θ is the rotation, and l is the length of each yield line in the assumed mechanism.

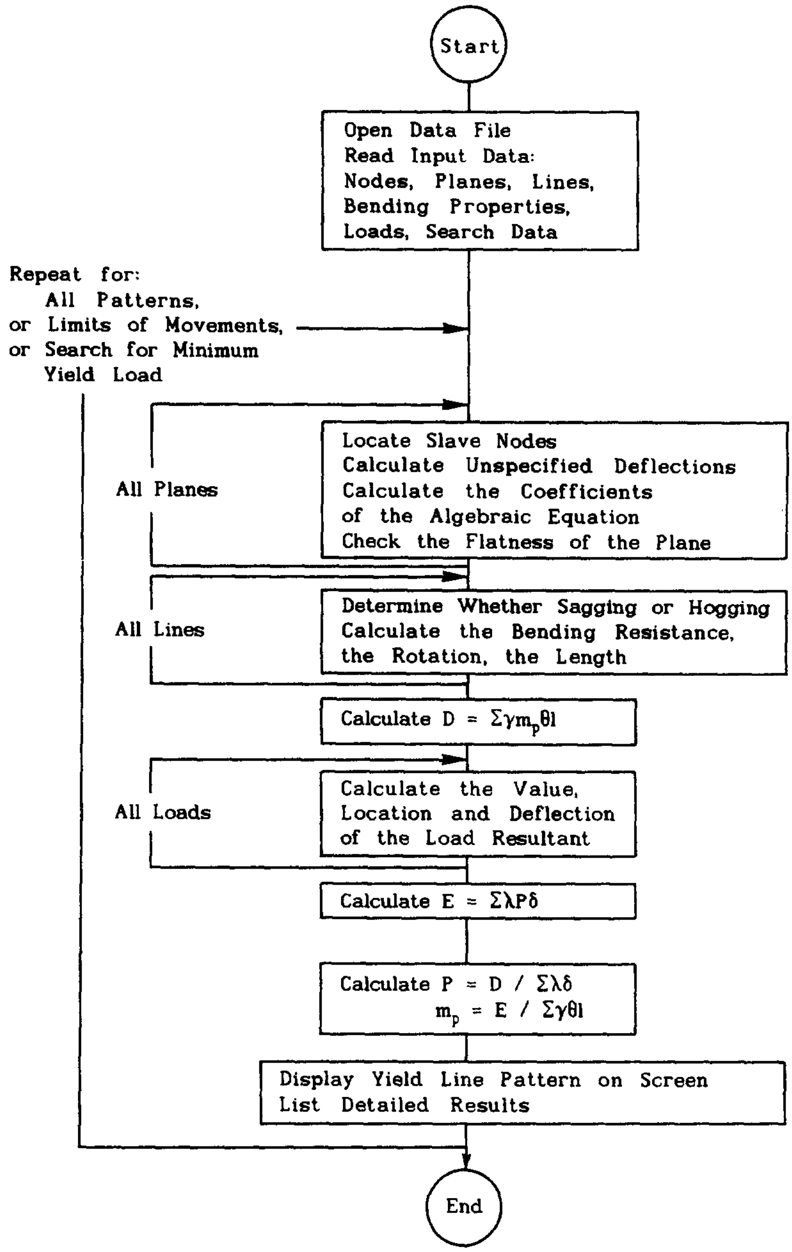

Fig. 1. Flow chart of the numerical yield line analysis program.

Since the yield line method leads to an upper bound solution, different mechanisms as well as different dimensions for each mechanism must be tried in order to find the lowest predicted strength of the plate. In the conventional method, the optimum solution of simple problems can be found directly by differentiation. For complex problems, a trial and error technique is faster and usually satisfactory [2, 3]. For the numerical method presented herein, a simple searching procedure is used to find the optimum solution.

NUMERICAL METHOD

The calculations involved in the numerical solution described below are lengthy and while straightforward and tractable by hand, computer implementation is necessary for practical use. A flow chart of a program is shown in Fig. 1, and its main features are discussed below.

The numerical method consists of computing the yield load or required bending resistance of a plate based on the geometry of an assumed mechanism defined by means of nodes, planes and lines. Consider, as a simple example, an orthotropic square plate with fixed supports, subjected to a point load P at the center and assumed to form the yield line mechanism shown in Fig. 2. The yield lines are numbered from 1 to 8 with end nodes numbered from

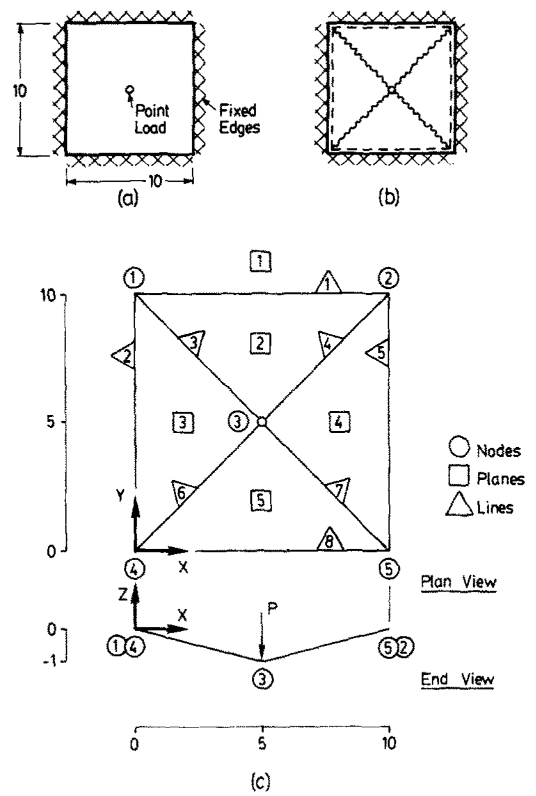

Fig. 2. Fixed edges square plate with central concentrated load: (a) plate; (b) yield line pattern; (c) model for numerical analysis.

1 to 5. The flat plate segments, or planes, are numbered from 1 to 5, including plane 1 which represents the plane containing the fixed supports. A right-hand rectangular coordinate system is set with the origin located arbitrarily, say at the lower left corner, with the z axis pointing upward. The (x_i, y_i, z_i) coordinates of each node are then determined.

The energy dissipated by the yield lines is discussed first. This includes the calculation of the plastic moment, the rotation and the length of the yield lines.

The bending resistance per unit length, m_p, of a yield line making an angle α with the x axis is, in an orthotropic plate (Fig. 3a), if the yield line is sagging

$$m_p = m_{px}\cos^2\alpha + m_{py}\sin^2\alpha \tag{5}$$

and if the yield line is hogging

$$m_p = m'_{px}\cos^2\alpha + m'_{py}\sin^2\alpha \tag{6}$$

where the functions of α are found from

$$\cos^2\alpha = \left[\frac{y_2 - y_1}{l}\right]^2, \quad \sin^2\alpha = \left[\frac{x_2 - x_1}{l}\right]^2. \tag{7}$$

m_{px} and m_{py} are the sagging resistances in the x and y direction, respectively, and m'_{px} and m'_{py} are the hogging resistances. x_1, y_1 and x_2, y_2 are the x and y coordinates of the end nodes of the yield line and l is the length of the yield line.

In skew concrete slabs the reinforcement may be placed parallel to the edges of the slab, and hence the plate is not orthotropic. Let the reinforcement be placed in the x direction and in the s direction, inclined at an angle β with the x axis ($0° < \beta < 180°$). The bending resistance, m_p, of a yield line making an angle α with the x axis is (Fig. 3b), if the yield line is sagging

$$m_p = m_{px}\cos^2\alpha + m_{ps}\cos^2(\beta - \alpha) \tag{8}$$

and if the yield line is hogging

$$m_p = m'_{px}\cos^2\alpha + m'_{ps}\cos^2(\beta - \alpha) \tag{9}$$

where the functions of α are found from

$$\cos^2\alpha = \left[\frac{y_2 - y_1}{l}\right]^2; \tag{10}$$

if $(y_2 - y_1)(x_2 - x_1) > 0$:

$$\cos^2(\beta - \alpha) = \left[\frac{(\cos\beta)(y_2 - y_1)}{l} + \frac{(\sin\beta)(|x_2 - x_1|)}{l}\right]^2 \tag{11}$$

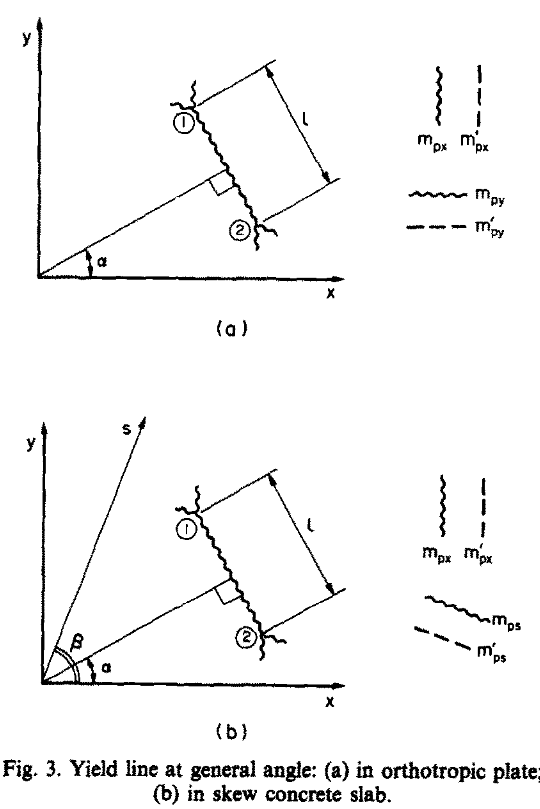

Fig. 3. Yield line at general angle: (a) in orthotropic plate; (b) in skew concrete slab.

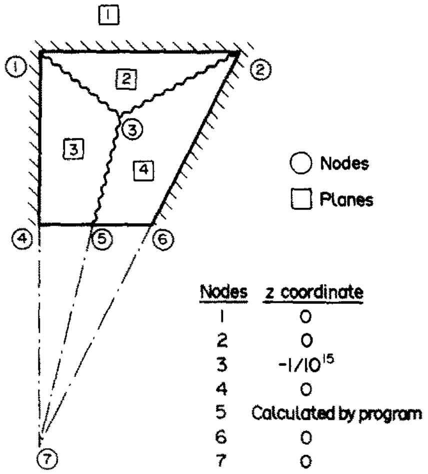

Fig. 4. Example where the z coordinate of node 5 can be calculated using the equation of plane 3 (or 4).

if $(y_2 - y_1)(x_2 - x_1) \leqslant 0$:

$$\cos^2(\beta - \alpha) = \left[\frac{(\cos\beta)(y_1 - y_2)}{l} + \frac{(\sin\beta)(|x_2 - x_1|)}{l}\right]^2. \quad (12)$$

m_{px}, m'_{px}, m_{ps} and m'_{ps} are the sagging and hogging resistances in the x and s direction, respectively.

Before calculating the rotation of the yield line, planes must be defined, as follows, corresponding to the rigid plate segments of the assumed mechanism. For the plate shown in Fig. 2, plane 2 is defined by nodes 1, 2 and 3, plane 3 is defined by nodes 1, 3 and 4, etc. Given three points $p_0(x_0, y_0, z_0)$, $p_1(x_1, y_1, z_1)$ and $p_2(x_2, y_2, z_2)$, the algebraic equation of the plane through these points is

$$Ax + By + Cz + D = 0 \quad (13)$$

where

$$A = (y_1 - y_0)(z_2 - z_0) - (z_1 - z_0)(y_2 - y_0)$$

$$B = (z_1 - z_0)(x_2 - x_0) - (x_1 - x_0)(z_2 - z_0)$$

$$C = (x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(x_2 - x_0)$$

$$D = -(Ax_0 + By_0 + Cz_0).$$

In order to define a plane, the three points p_0, p_1 and p_2 must not be colinear. This can be checked by comparing the slopes of a line from p_0 to p_1 and a line from p_1 to p_2. For simplicity, the slopes in the x, y plane, $(y_1 - y_0)/(x_1 - x_0)$ and $(y_2 - y_1)/(x_2 - x_1)$ are compared. If the slopes are unequal the three points are not colinear and can be used to calculate the algebraic equation of the plane.

Once the equation of a plane has been determined, it can be used to calculate the deflection of some nodes, which otherwise would have to be calculated by hand. Figure 4 shows an example where the z coordinate of node 5 can be calculated using the equation of planes 3 or 4. The equation defining the plane of a segment can also be used to check whether the segment is indeed planar. This is done by comparing the z coordinates of each node to the value calculated based on the x and y coordinates of the node and using the equation of the plane, i.e.

$$z = -\frac{Ax + By + D}{C}. \quad (14)$$

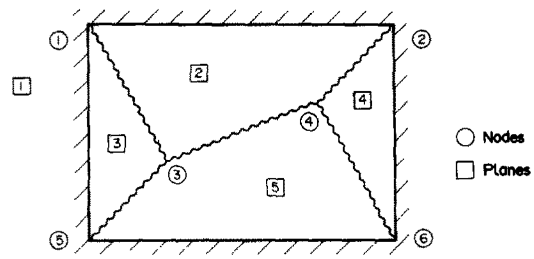

Fig. 5. Example of a mechanism that is not kinematically admissible.

The mechanism shown in Fig. 5 is not kinematically admissible. This is detected when checking the z coordinates of nodes 3 and 4 on the contiguous rigid plate segments 2 and 5.

The rotation of each yield line is given by the angle θ between the two planes intersecting at that yield line (see Fig. 6). Given two planes m and n with the following algebraic equations

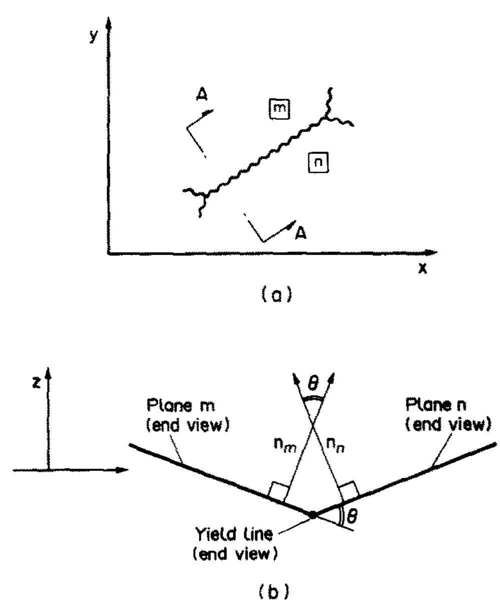

Fig. 6. Rotation between the rigid plate segments: (a) plan view; (b) section A–A.

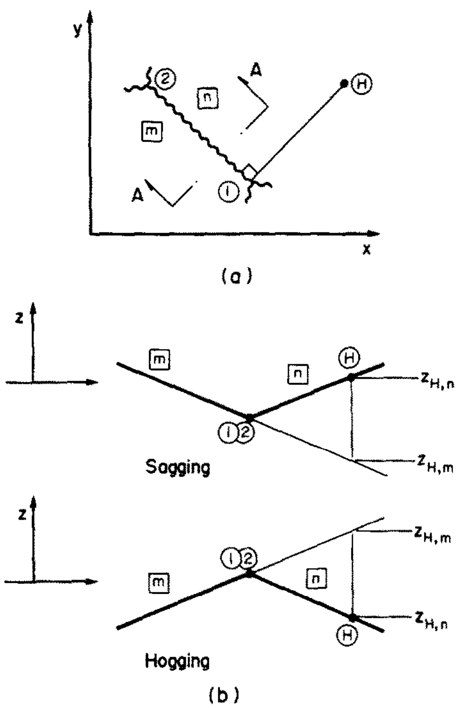

Fig. 7. Bending sign (sagging or hogging) of yield line: (a) plan view; (b) section A–A.

$$\text{plane } m: \quad A_m x + B_m y + C_m z + D_m = 0$$

$$\text{plane } n: \quad A_n x + B_n y + C_n z + D_n = 0 \qquad (15)$$

the angle θ between these planes is equal to the acute angle between their normal vectors $\mathbf{n}_m$ and $\mathbf{n}_n$ and is given by

$$\theta \cong \tan\theta = \frac{|\mathbf{n}_m \times \mathbf{n}_n|}{|\mathbf{n}_m \cdot \mathbf{n}_n|} = \frac{\sqrt{(B_m C_n - C_m B_n)^2 + (C_m A_n - A_m C_n)^2 + (A_m B_n - B_m A_n)^2}}{|A_m A_n + B_m B_n + C_m C_n|} \qquad (16)$$

where, since we consider virtual displacements, the angle can be considered small. Such small angles are obtained by assuming small deflections of the yield line mechanism. For example, choosing a maximum value of $1/10^{15}$ of the plate width, say, for the z coordinate of the nodes in the displaced plate leads to satisfactory results with less than $1/10^{10}\%$ error.

From the numerical description of a yield line mechanism, it is possible to determine the bending sign of the yield lines, i.e. whether they are sagging or hogging. Given a yield line with end nodes 1 and 2, bounded by planes m and n, and using the convention that plane m is on the left-hand side of the yield line for an observer standing at node 1 and looking at node 2, then a point H with coordinates $[x_1 + (y_2 - y_1), y_1 + (x_1 - x_2)]$ is always on the right-hand side of the yield line (see Fig. 7). The difference between the z coordinate of point H on plane n and the corresponding coordinate using the equation of plane m indicates whether the yield line is sagging or hogging. When $z_{H,n} - z_{H,m} > 0$, the yield line is sagging. When $z_{H,n} - Z_{H,m} < 0$, the yield line is hogging.

The length of each yield line is given by the distance between its end nodes $p_1(x_1, y_1, z_1)$ and $p_2(x_2, y_2, z_2)$, and is equal to

$$\overline{p_1 p_2} = (x_2 - x_1)^2 + (y_2 - y_1)^2 \qquad (17)$$

where the z coordinates, being very small, are not included.

The plastic moment, the rotation and the length of each yield line having been found using eqns (5)–(17), the product of these values is then summed for all yield lines. The sum, $\Sigma \gamma m_p \theta l$, is equal to the total energy, D, dissipated by the yield lines.

The work done by the loads is now discussed. Point loads, line loads, and uniformly distributed loads (UDL) are treated.

Point loads can be defined by a magnitude λ_{PL}, a point (node) with coordinates x and y where the load is acting, and the plane on which it is acting. The deflection of the load is the z coordinate of the point where the load is acting. If the z coordinate is not specified at the load point, it can be calculated from eqn (14). The work done by the point load is then

$$E_{PL} = \lambda_{PL} P z_{PL}. \qquad (18)$$

A uniform or linearly varying line load can be defined by two end nodes with coordinates x_1, y_1 and x_2, y_2, the magnitude of the line load at each end, λ_1 and λ_2, and the plane on which the line load is applied (see Fig. 8). Given this data, the work done by the line load is calculated as follows.

The length of the line load is

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (19)$$

The resultant of the line load is

$$\lambda_{LL} = \frac{\lambda_1 + \lambda_2}{2} l. \qquad (20)$$

The location of the resultant is at (x_c, y_c) where

$$x_c = x_1 + \frac{x_2 - x_1}{l} c, \quad y_c = y_1 + \frac{y_2 - y_1}{l} c, \qquad (21)$$

where

$$c = \frac{2\lambda_2 + \lambda_1}{3(\lambda_2 + \lambda_1)} l. \qquad (22)$$

The deflection at the point through which the resultant load acts is

$$z_{LL} = -(Ax_c + By_c + D)/C \qquad (23)$$

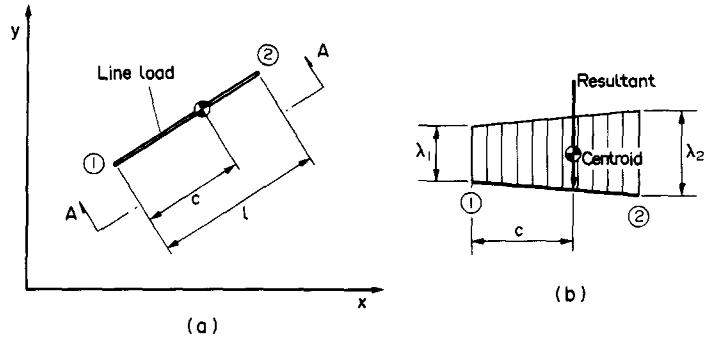

Fig. 8. Line load.

where A, B, C and D are the coefficients of the algebraic equation of the plane on which the load is acting.

Finally, the work done by the line load is

$$E_{\mathrm{LL}} = \lambda_{\mathrm{LL}} P z_{\mathrm{LL}}. \tag{24}$$

A UDL can be defined by the n vertices, with coordinates (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), of the area covered by the UDL, the magnitude of the UDL, λ_{UDL}, and the plane on which it is applied. Given these data, the work done by the load is calculated as follows.

The value and the location of the resultant are calculated in a manner similar to that by which the area of a traverse is calculated in surveying (see Fig. 9). The unit resultant load is

$$P_x = (y_2 - y_1)(x_1 + x_2)/2 + (y_3 - y_2)(x_2 + x_3)/2 + \cdots + (y_1 - y_n)(x_n + x_1)/2; \tag{25}$$

alternatively,

$$P_y = (x_2 - x_1)(y_1 + y_2)/2 + (x_3 - x_2)(y_2 + y_3)/2 + \cdots + (x_1 - x_n)(y_n + y_1)/2. \tag{26}$$

The location of the resultant is at the centroid of the area covered by the UDL, i.e. at $[x_c, y_c]$ where

$$x_c = \frac{\Sigma \bar{x} p}{P_y}, \quad y_c = \frac{\Sigma \bar{y} p}{P_x}, \tag{27}$$

where

$$\Sigma \bar{x} p = \frac{y_2 - y_1}{8}\left[(x_1 + x_2)^2 + \frac{(x_2 - x_1)^2}{3}\right] + \frac{y_3 - y_2}{8}\left[(x_2 + x_3)^2 + \frac{(x_3 - x_2)^2}{3}\right] + \cdots + \frac{y_1 - y_n}{8}\left[(x_n + x_1)^2 + \frac{(x_1 - x_n)^2}{3}\right] \tag{28}$$

and

$$\Sigma \bar{y} p = \frac{x_2 - x_1}{8}\left[(y_1 + y_2)^2 + \frac{(y_2 - y_1)^2}{3}\right] + \frac{x_3 - x_2}{8}\left[(y_2 + y_3)^2 + \frac{(y_3 - y_2)^2}{3}\right] + \cdots + \frac{x_1 - x_n}{8}\left[(y_n + y_1)^2 + \frac{(y_1 - y_n)^2}{3}\right]. \tag{29}$$

The deflection at the resultant is

$$z_{\mathrm{UDL}} = -(Ax_c + By_c + D)/C, \tag{30}$$

where A, B, C and D are the coefficients of the algebraic equation of the plane on which the load is acting.

Finally the work done by the UDL is

$$E_{\mathrm{UDL}} = \lambda_{\mathrm{UDL}} P z_{\mathrm{UDL}}. \tag{31}$$

Note that the above procedure to determine the work done by a UDL is valid for UDLs covering areas of arbitrary polygonal shape, including areas with re-entrant corners and with holes.

The work done by the various loads having been found using eqns (18)–(31), the values are then

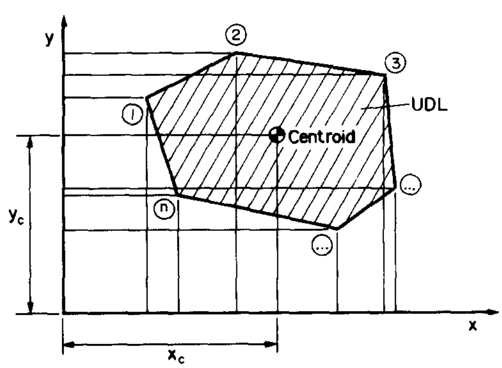

Fig. 9. Uniformly distributed load (UDL).

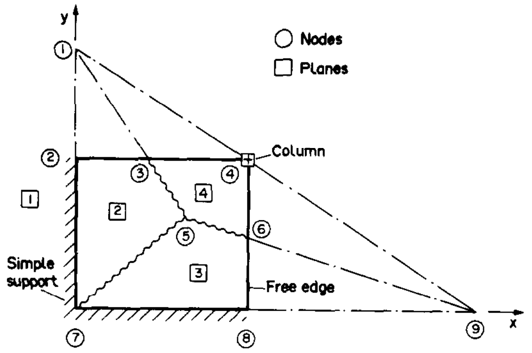

Fig. 10. Example where nodes 3 and 6 are set as slave nodes.

summed for all the loads. This sum, $\Sigma\lambda P\delta$, is equal to the total work done by the loads, E.

Finally, the yield load P of the plate is found by using eqn (3), i.e. by dividing E by the sum of $\lambda\delta$. For the example in Fig. 2, with $\lambda = 1$ and all $\gamma m_p = 1$ ($\gamma m_{px} = \gamma m_{py} = \gamma m'_{px} = \gamma m'_{py} = 1$, i.e. an isotropic plate), the value obtained for P is 16. Alternatively, by using eqn (4), the bending resistance of the plate, m_p, required to support a load of $\lambda P = 1$ is found to be 0.0625. These results agree with the solution from a usual yield line analysis of this mechanism.

Mechanisms involving fans (curved yield lines) can be treated by approximating the fan using a series of triangles placed one next to the other. Using 16 such triangles, a complete circular fan is approximated with a 1.3% error. Fans of elliptical, logarithmic, etc. shape can be approximated in a similar manner. Slabs on beams and slabs with curtailed reinforcement can be treated by assigning the appropriate bending resistance to yield lines or segments of yield lines corresponding to the beams and to the actual placement of the reinforcement.

To find the optimum solution for a yield line mechanism, a series of patterns can be defined and the yield load calculated for each pattern. The pattern giving the minimum yield load is retained as the solution.

Series of patterns can be produced by specifying, for one or more nodes, initial and final positions in the x,y plane, and the number of steps between these positions. These values can be used by an iteration procedure which creates the family of patterns. When a yield line mechanism involves several parameters, a corresponding number of iteration procedures can be used to generate all the families of patterns. The iteration procedures should preferably be nested, so that all possible patterns are created in the same solution. Recursive procedures can be used advantageously for this purpose.

When generating a series of yield line patterns, it is often possible to relate the location of some of the moving nodes to the location of other nodes. This reduces the amount of data required for the optimization, and also it conveniently restricts the movements of the nodes within the limit of validity for the mechanism. One way of establishing the relationship is by locating a node at the intersection of two lines defined by two pairs of nodes. The node at the intersection is called a slave node, while the other four nodes guiding the slave node are called master nodes. Figure 10 shows an example where nodes 3 and 6 are slave nodes with master nodes 1, 5 and 2, 4 and 4, 8 and 5, 9, respectively. If a family of patterns is created by moving node 5, then the location of nodes 3 and 6 is adjusted automatically.

COMPUTER PROGRAM

The program is menu driven. It starts by opening the required data file. If the data specifies a search for the optimum solution, the user can choose between

—analysing all the patterns created by the search;
—analysing only the patterns with the initial and final node positions of each movement;
—searching directly for the minimum yield load (or maximum required bending resistance).

The yield line patterns are displayed on the screen, and nodes deflecting upwards, downwards and not deflecting are each shown differently. Sagging, hogging and construction lines are indicated by bright solid lines, bright dash lines and light lines, respectively. Point loads, line loads and UDLs are also shown. The calculated yield load, or the required bending resistance value, is displayed at the bottom of the screen.

For each pattern analysed, detailed results can be listed by the program. These results include:

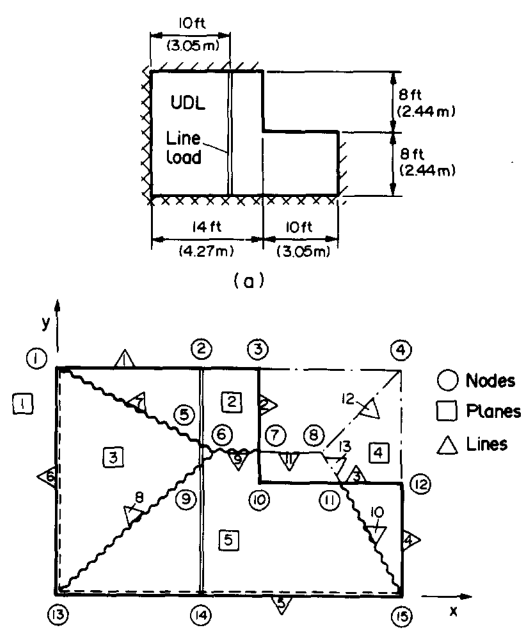

Fig. 11. Reinforced concrete corner panel of floor system: (a) slab; (b) model for numerical analysis (from [3]).

Table 1. Example. Listing of results

=========================
Yield Line Analysis Program
=========================

DATA FILE : EXAMPLE.DAT
===========

Z-COORDINATE FACTOR = 1.00000000000000E-015
==================

NODES
=====

NO.	COORDINATES X	Y	Z	
1	0.000	16.000	0.000	
2	10.000	16.000	0.000	
3	14.000	16.000	0.000	
4	24.000	16.000	0.000	
5	10.000	10.000	-0.909	(Z Computed)
6	11.000	9.400	-1.000	
7	14.000	9.400	-1.000	(Z Computed)
8	17.333	9.400	-1.000	
9	10.000	8.545	-0.909	(Z Computed)
10	14.000	8.000	-0.851	(Z Computed)
11	18.326	8.000	-0.851	(Z Computed)
12	24.000	8.000	0.000	
13	0.000	0.000	0.000	
14	10.000	0.000	0.000	
15	24.000	0.000	0.000	

PLANES
======

NO.	THROUGH THESE NODES P0	P1	P2	etc.					
1	1	2	3	4	12	15	14	13	
2	1	2	3	4	5	6	7	8	
3	1	5	6	9	13				
4	4	8	11	12	15				
5	6	7	8	9	10	11	13	14	15

PLANES
======

NO.	THROUGH THESE 3 NODES			ALGEBRAIC EQUATION A	B	C	D
1	1	2	12	0.000000	0.000000	-80.000000	0.000000
2	1	2	6	0.000000	1.00000E-014	-66.000000	-1.60000E-013
3	1	5	13	-1.45455E-014	0.000000	-160.000000	0.000000
4	4	8	12	-8.00000E-015	0.000000	53.333333	1.92000E-013
5	6	7	9	0.000000	-2.72727E-016	-2.563636	2.36658E-030

LINES
=====

NO.	BETWEEN THESE 2 POINTS		BETWEEN THESE 2 PLANES		LENGTH	ROTATION (RAD)	BENDING SIGN	BRN	MP	ENERGY DISSIPATED
1	1	4	1	1	24.000000	0.000000	no work	1	0.000000	0.000000
2	3	10	1	1	8.000000	0.000000	no work	1	0.000000	0.000000
3	10	12	1	1	10.000000	0.000000	no work	1	0.000000	0.000000
4	4	15	1	1	16.000000	0.000000	no work	1	0.000000	0.000000
5	13	15	5	1	24.000000	0.106383	Hogging	1	1.000000	2.553191
6	13	1	1	3	16.000000	0.090909	Hogging	1	1.000000	1.454545
7	1	6	2	3	12.828094	0.176696	Sagging	1	1.000000	2.266667
8	13	6	3	5	14.469278	0.139935	Sagging	1	1.000000	2.024758
9	6	7	2	5	3.000000	0.257898	Sagging	1	1.000000	0.773694
10	15	11	5	4	9.807729	0.183895	Sagging	1	1.000000	1.803591
11	7	8	1	1	3.333333	0.000000	no work	1	0.000000	0.000000
12	8	4	1	1	9.381068	0.000000	no work	1	0.000000	0.000000
13	8	11	1	1	1.716352	0.000000	no work	1	0.000000	0.000000

Table 1 (*contd*)

```
BENDING RESISTANCE PROPERTIES
=============================
 NO.

  1  ORTHOTROPIC
     SAGGING:  MX=      1.000000       MY=      1.000000
     HOGGING:  MX=      1.000000       MY=      1.000000


POINT LOADS
===========
WORK DONE BY ALL POINT LOADS =      0.000

 NO POINT LOADS


LINE LOADS
==========
WORK DONE BY ALL LINE LOADS =   4046.281

LINE  ON    --------------- FROM ---------------   ----------------- TO ----------------
LOAD  PLANE      LOAD            COORDINATES           LOAD            COORDINATES
 NO.             FACTOR          X          Y          FACTOR          X          Y

  1    2     -510.000000     10.000     16.000     -510.000000     10.000     10.000
  2    3     -510.000000     10.000     10.000     -510.000000     10.000      8.545
  3    5     -510.000000     10.000      8.545     -510.000000     10.000      0.000


LINE  ON    --------------- CENTROID -------------    TOTAL      WORK
LOAD  PLANE       COORDINATES         DISPLACEMENT    LOAD       DONE
 NO.              X           Y           (Z)

  1    2       10.000      13.000     -0.454545   -3060.000   1390.909
  2    3       10.000       9.273     -0.909091    -741.818    674.380
  3    5       10.000       4.273     -0.454545   -4358.182   1980.992

UNIFORMLY DISTRIBUTED LOADS
===========================
 WORK DONE BY ALL UDLs =  34183.983

 UDL  ON        LOAD     --------------- CENTROID -------------     TOTAL       WORK
 NO.  PLANE     FACTOR          COORDINATES        DISPLACEMENT      LOAD        DONE
                              X           Y            (Z)
  1    2     -310.000000     9.157     13.412     -0.392157  -17391.000   6820.000
  2    3     -310.000000     3.667      8.467     -0.333333  -27280.000   9093.333
  3    5     -310.000000    12.754      3.597     -0.382634  -42533.539  16274.774
  4    4     -310.000000    22.109      5.333     -0.283688   -7035.461   1995.875

FINAL RESULTS
=============

  D = ENERGY DISSIPATED BY THE YIELD LINES  =    10.8764475904

  E = WORK DONE BY ALL THE LOADS            = 38230.2636887830

  P = YIELD LOAD OF THE STRUCTURE = D / E   =     0.0002844984

  mp = PLASTIC MOMENT RESISTANCE
        / UNIT LENGTH          = E / D   =  3514.9586637461
```

—an echo of the input data;

—the coefficients of the algebraic equation of the planes;

—the bending sign (sagging or hogging), the plastic resistance m_p [from eqns (5), (6), (8) or (9)], the rotation, the length and the energy dissipated by each yield line;

—for the point loads, for the line loads, and for the UDLs: the value, the location, and the displacement of the resultant load, and the work done by each load;

—the total energy dissipated by the yield lines, D, the work done by all the loads, E, the yield load of the plate structure for the assumed mechanism,

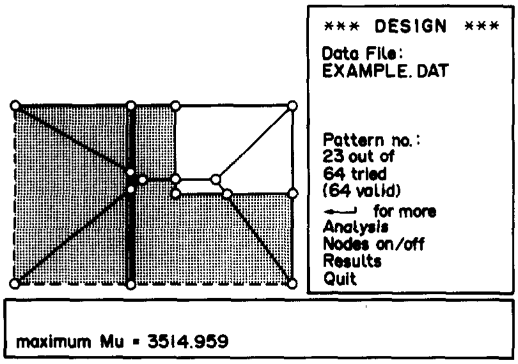

Fig. 12. Reinforced concrete corner panel of floor system. Screen display from the yield line analysis program.

P, and the required bending resistance of the plate, m_p.

Further details about the input data and the use of the program are given in a users' manual [5].

EXAMPLE

A corner panel of a floor system is continuous with adjacent panels at supporting beams along two edges and simply supported at the other edges except for a rectangular opening that is unsupported at its edges. Figure 11(a) shows the panel. The service loads are a uniformly distributed live load of 100 psf (4.78 kN/m^2) and a line load of 300 lb/ft (4.38 kN/m). This example is taken from [3, example 8.6].

The numerical model of the slab is shown in Fig. 11(b). Figure 12 and Table 1 show the screen display and the output listing, respectively, for the yield line pattern with the maximum required bending resistance of the slab. The result agrees with that given in [3].

CONCLUDING REMARKS

A numerical method based on the virtual work method of the yield line theory has been described, and a program has been written which uses this method to analyse plate structures and which can treat features such as orthotropy and skewness, point loads, line loads, uniformly distributed loads, fans, etc. The program also includes procedures for the optimization of the yield line mechanisms. Since the method is numerical, it allows the yield line analysis of plates with complex shapes, assumed mechanisms and loadings.

Acknowledgements—The authors thank McGill University and the Natural Sciences and Engineering Research Council (Grant A-3366) for supporting this project.

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